Ordering results for elliptical distributions with applications to risk bounds

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Abstract

A classical result of Slepian [32] for the normal distribution and extended by Das Gupta et al. [14] for elliptical distributions gives one-sided (lower orthant) comparison criteria for the distributions with respect to the (generalized) correlations. Müller and Scarsini [24] established that the ordering conditions even characterize the stronger supermodular ordering in the normal case. In the present paper, we extend this result to elliptical distributions. We also derive a similar comparison result for the directionally convex ordering of elliptical distributions. As application, we obtain several results on risk bounds in elliptical classes of risk models under restrictions on the correlations or on the partial correlations. Furthermore, we obtain extensions and strengthenings of recent results on risk bounds for various classes of partially specified risk factor models with elliptical dependence structure of the individual risks and the common risk factor. The moderate dependence assumptions on this type of models allow flexible applications and, in consequence, are relevant for improved risk bounds in comparison to the marginal based standard bounds.

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1. Introduction

In the first part of this paper, we extend and strengthen some basic stochastic ordering results for multivariate normal distributions to the frame of elliptical distributions. As a consequence, we obtain in the second part of the paper upper bounds in classes of elliptical distributions under restrictions on (partial) correlations as well as extensions and strengthenings of several recent results on risk bounds in partially specified factor models (PSFM's) with elliptical specification of the dependence structure of the individual risks with a common risk variable.

A classical result of Slepian [32, Lemma 1.1] gives one-sided (lower orthant) comparison criteria of normal distributions by the increase of the off-diagonal correlations. In the paper of Block and Sampson [11, Theorem 2.1 and Corollary 2.3] it is stated that an increase of the off-diagonal correlations even implies the supermodular comparison of these distributions. The argument in [11] is shown in Müller and Scarsini [24, Section 4] to be incomplete. In their paper these authors give a complete proof of the strengthened comparison result for the normal case. In the present paper, we use the ideas in these papers to characterize the supermodular ordering for the general elliptical case. We also derive a related ordering criterion for the comparison of elliptical distributions w.r.t. (with respect to) the directionally convex order.

The ordering results are used in Sections 3 and 4 of the paper to derive unique worst case distributions for several relevant classes of risk models. These include models with elliptical dependence structure and additional bounds on correlations and partial correlations corresponding to a C-vine structure. We also show that a generalization to D-vine structures and, thus, to arbitrary regular vine structures is not possible. A second type of applications concerns PSFM's which do not need a full specification of the dependence structure and, thus, are a particular flexible tool for applications. Under various constraints on the specifications, i.e., on the dependence structure of the individual risks

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with the risk factor, unique worst case distributions are determined. As consequence, these results imply relevant improvements of standard (upper) risk bounds based only on marginal information on the risk vectors.

2. Supermodular and directionally convex order in classes of elliptical distributions

In this section, we characterize the supermodular ordering and the directionally convex ordering in classes of elliptical distributions with a fixed generator. For the directionally convex ordering, also an extension to the case of different generators is given. Since comparison w.r.t. the supermodular ordering needs identical marginals, an extension of this form for the supermodular order is not possible.

For a function $f\colon \mathbb{R}^d \to \mathbb{R}$, let $\Delta_i^\varepsilon f(x) := f(x + \varepsilon e_i) - f(x)$ be the difference operator, where $\varepsilon > 0$ and where e_i, $i = 1, \ldots, d$, denote the unit vectors w.r.t. the canonical base in $\mathbb{R}^d$. Then, f is said to be supermodular, respectively, directionally convex if $\Delta_i^{\varepsilon_i} \Delta_i^{\varepsilon_j} f \geq 0$ for all $1 \leq i < j \leq d$, respectively, $1 \leq i \leq j \leq d$. For d-dimensional random vectors ξ, ξ', the supermodular order $\xi \leq_{sm} \xi'$, respectively, the directionally convex order $\xi \leq_{dcx} \xi'$ is defined via $\mathbb{E} f(\xi) \leq \mathbb{E} f(\xi')$ for all supermodular, respectively, directionally convex functions f for which the expectations exist. The lower orthant order $\xi \leq_{lo} \xi'$ is defined by the pointwise comparison of the corresponding distribution functions, i.e. $F_\xi(x) \leq F_{\xi'}(x)$ for all $x \in \mathbb{R}^d$. Remember that the stochastic order $\zeta \leq_{st} \zeta'$, respectively, the convex order $\zeta \leq_{cx} \zeta'$ for real-valued random variables ζ, ζ' is defined via $\mathbb{E}\varphi(\zeta) \leq \mathbb{E}\varphi(\zeta')$ for all increasing, respectively, convex functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ for which the expectations exist.

For an overview of stochastic orderings, see Müller and Stoyan [26], Shaked and Shantikumar [31], and Rüschendorf [28].

A d-dimensional random vector X has an elliptically contoured (or, shortened, elliptical) distribution with parameters μ, Σ, and generator ϕ, written

$$X \sim \mathcal{E}C_d(\mu, \Sigma, \phi),$$

if $\mu \in \mathbb{R}^d$, Σ is a $d \times d$ positive semi-definite symmetric matrix, and if the characteristic function $\varphi_{X-\mu}$ of $X - \mu$ is a function of the quadratic form $t \Sigma t^\top$, i.e., $\varphi_{X-\mu}(t) = \phi(t \Sigma t^\top)$.

For $k = \operatorname{rank}(\Sigma)$, elliptical random vectors have a characterization by a stochastic representation of the form

$$X \overset{\mathrm{d}}{=} \mu + R_k U^{(k)} A, \tag{1}$$

where $U^{(k)}$ is a random vector (of dimension k) which is uniformly distributed on the unit sphere in $\mathbb{R}^k$, where the radial variable $R_k = R_{k,\phi}$ is a non-negative random variable independent of $U^{(k)}$, and where A is a deterministic $k \times d$ matrix such that $\Sigma = A^\top A$, see Cambanis et al. [12].

A necessary and sufficient condition for such a representation is that $\phi \in \Phi_\ell$ for some $\ell \geq k$, where $\Phi_\ell \subset \Phi_k$ denotes the class of functions $\psi\colon [0, \infty) \to \mathbb{R}$ with

$$\psi(u) = \int_{[0,\infty)} \Omega_\ell(r^2 u) \, \mathrm{d}G_\ell(r) \tag{2}$$

for Ω_ℓ being the characteristic function of $U^{(\ell)}$ and G_ℓ the distribution function of R_ℓ, see Cambanis et al. [12, Corollary 2]. The relationship between $\psi \in \Phi_\ell$ and G_ℓ is one-to-one, see [12, Theorem 1].

Exactly in the case that $\mathbb{E} R_k^2 < \infty$, or equivalently that $\phi'(0)$ is finite, the covariance matrix exists and is proportional to Σ (see Fang and Zhang [16, p. 67]).

A well-known property of elliptical distributions is that they are closed under marginalization where the marginals inherit the elliptical generator (but not the radial variable), see, e.g., Fang and Zhang [16, Corollary 1 of Theorem 2.6.3] and Cambanis et al. [12, Corollary 2].

Further, elliptical distributions are closed under conditioning, see Cambanis et al. [12, Corollary 5]. In contrast to the marginalization property, the generator is not necessarily inherited.

2.1. Supermodular ordering of elliptical distributions

The lower orthant ordering of multivariate normal distributions with fixed univariate marginal distributions goes back to Slepian [32, Lemma 1.1] (see also Tong [34]). An extension to elliptical distributions is established in Das Gupta et al. [14, Theorem 5.1]. In the bivariate case, this is equivalent to the supermodular ordering.

The following special comparison result is given in Block and Sampson [11, Theorem 2.1 and Lemma 2.2], see also Müller and Scarsini [24, Lemma 4.1]. The proof is based essentially on a conditioning argument leading to a reduction to a comparison of two-dimensional elliptical distributions. We give a sketch of the proof since we make use of some arguments of it later on in the paper.

Lemma 1. *Let $X \sim \mathcal{E}C_d(\mu, \Sigma, \phi)$ and $Y \sim \mathcal{E}C_d(\mu, \Sigma', \phi)$ with $\sigma_{ij} \leq \sigma'_{ij}$ and $\sigma_{k\ell} = \sigma'_{k\ell}$ for all $(k, \ell) \notin \{(i, j), (j, i)\}$ for some $i \neq j$. Then, $X \leq_{sm} Y$.*

Proof. Let $X = (X_1, \dots, X_d)$ and $Y = (Y_1, \dots, Y_d)$. Assume without loss of generality that $(i, j) = (1, 2)$. In the first case, assume that both Σ and Σ' are positive definite matrices. Write

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \tag{3}$$

where Σ_{11} is the two-dimensional (generalized) covariance matrix of (X_1, X_2) and Σ_{22} denotes the $(d-2)$-dimensional (generalized) covariance matrix of $(X_3, \dots, X_d)$. Partition Σ' and $\mu = (\mu_1, \mu_2)$ in the same way.

For $z \in \mathbb{R}^{d-2}$, let $\mu_z = \mu_1 + (z - \mu_2)\Sigma_{22}^{-1}\Sigma_{21}$,

$$\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\,, \quad \text{and} \quad \Sigma'_{11.2} = \Sigma'_{11} - \Sigma'_{12}\Sigma_{22}'^{-1}\Sigma'_{21}\,.$$

Then, for the conditional distributions holds

$$(X_1, X_2) \,|\, (X_3, \dots, X_d) = z \ \sim \mathcal{E}C_2(\mu_z, \Sigma_{11.2}, \phi_{q(z)}) \quad \text{and} \quad (Y_1, Y_2) \,|\, (Y_3, \dots, Y_d) = z \ \sim \mathcal{E}C_2(\mu_z, \Sigma'_{11.2}, \phi_{q(z)})\,, \tag{4}$$

for some generator $\phi_{q(z)}$ depending only on ϕ and $q(z) = (z - \mu_2)\Sigma_{22}^{-1}(z - \mu_2)^\top$, see Cambanis et al. [12, Corollary 5]. Thus, the conditional distribution depends on Σ_{11} and Σ'_{11} only through $\Sigma_{11.2}$ and $\Sigma'_{11.2}$, respectively.

Since by assumption it holds componentwise that $\Sigma_{11} \leq \Sigma'_{11}$, $\Sigma_{12} = \Sigma'_{12}$, $\Sigma_{21} = \Sigma'_{21}$ and $\Sigma_{22} = \Sigma'_{22}$, it follows that $\Sigma_{11.2} \leq \Sigma'_{11.2}$ componentwise with equality for the diagonal elements. Hence, the characterization of the supermodular ordering in the bivariate case implies

$$(X_1, X_2) \,|\, (X_3, \dots, X_d) = z \ \leq_{sm} \ (Y_1, Y_2) \,|\, (Y_3, \dots, Y_d) = z$$

for almost all z. Then, the concatenation property of the supermodular ordering yields $X \,|\, (X_3, \dots, X_d) = z \ \leq_{sm} Y \,|\, (Y_3, \dots, Y_d) = z$ for almost all z. Since $(X_3, \dots, X_d) \stackrel{\mathrm{d}}{=} (Y_3, \dots, Y_d)$, the statement follows from the closure of the supermodular ordering under mixtures, see Shaked and Shanthikumar [30, Theorem 2.4.].

In the second case assume that at least one of Σ and Σ' is positive semi-definite and not positive definite. Denote by I the identity matrix. Then, the matrices $\Sigma + \frac{1}{n}I$ and $\Sigma' + \frac{1}{n}I$ are positive definite for all $n \in \mathbb{N}$. According to the first case holds for $X_n \sim \mathcal{E}C_d(\mu, \Sigma + \frac{1}{n}I, \phi)$ and $Y_n \sim \mathcal{E}C_d(\mu, \Sigma' + \frac{1}{n}I, \phi)$ that $X_n \leq_{sm} Y_n$ for all $n \in \mathbb{N}$. Then, the statement follows from the closure of the supermodular ordering under weak convergence (see Müller and Scarsini [24, Theorem 3.5]). □

In the following theorem, we establish that the supermodular ordering of elliptical distributions is characterized by the componentwise ordering of the off-diagonal elements of the (generalized) covariance matrix. This result is the positive answer to the question formulated in Landsman and Tsanakas [21, Remark 2] whether the supermodular ordering results for multivariate normal distributions (see Müller [22, Theorem 11] and for Kotz-type distributions (see Ding and Zhang [15, Theorem 3.11]) can be extended to elliptical distributions of arbitrary dimension.

Theorem 1 ($\leq_{sm}$-ordering of elliptical distributions).
Let $X \sim \mathcal{E}\mathcal{C}_d(\mu, \Sigma, \phi)$ and $Y \sim \mathcal{E}\mathcal{C}_d(\mu', \Sigma', \phi')$ with $\Sigma = (\sigma_{ij})_{1\leq i,j\leq d}$, $\Sigma' = (\sigma'_{ij})_{1\leq i,j\leq d}$. Then, the following statements are equivalent:

(i) $X \leq_{sm} Y$,

(ii) $\mu = \mu'$, $\phi = \phi'$, $\sigma_{ii} = \sigma'_{ii}$ *f.a.* $1 \leq i \leq d$, *and* $\sigma_{ij} \leq \sigma'_{ij}$ *for all* $i \neq j$.

(iii) *X and Y have the same univariate marginals and* $\sigma_{ij} \leq \sigma'_{ij}$ *f.a.* i, j.

Proof. (i) $\Longrightarrow$ (iii): The supermodular ordering is a pure dependence ordering. Thus, the univariate marginal distributions must be equal and, in particular, $\sigma_{ii} = \sigma'_{ii}$ for all i. Since both the supermodular ordering and elliptical distributions are closed under marginalization, $\sigma_{ij} \leq \sigma'_{ij}$ follows from $(X_i, X_j) \leq_{lo} (Y_i, Y_j)$ with Landsman and Tsanakas [21, Corollary 2].
(iii) $\Longrightarrow$ (ii): This follows from the fact that elliptical distributions are closed under marginalization and the marginals inherit the elliptical generator.

Assume (ii). Consider two cases. In the first case let us assume that both matrices Σ and Σ' are positive definite. In the same way as in the proofs of Das Gupta et al. [14, Theorem 5.1] and Müller and Scarsini [24, Theorem 4.2.] there exists a finite sequence $\Sigma = \Sigma_1 \leq \cdots \leq \Sigma_k = \Sigma'$ (componentwise) of positive semi-definite matrices such that $\Sigma_{\ell+1}$ is obtained from Σ_ℓ by increasing exactly one off-diagonal entry. Hence, statement (i) follows from Lemma 1 and from the transitivity of the supermodular ordering.

In the second case assume that at least one of Σ and Σ' is positive semi-definite and not positive definite. Then, the statement follows from the first part and a similar approximation argument as in the second part of the proof of Lemma 1. □

Remark 1. The supermodular ordering result in Theorem 1 is established independently in a recent paper by Yin [35, Theorem 3.4] submitted to arXiv on Oct 16, 2019. For the proof, this author extends the integral representation argument in Müller [22, Theorem 11] in the normal case. We remark that our paper is based on the dissertation of the first author from Apr 09, 2019, see Ansari [3, Theorem 5.2], where Theorem 1 is given in explicit form.

2.2. Directionally convex ordering of elliptical distributions

In this section, we derive an ordering result for the directionally convex ordering of elliptical distributions. The following lemma is based on a conditioning argument similar to that used in the proof of Lemma 1.

Lemma 2. *Let $X \sim \mathcal{E}\mathcal{C}_d(\mu, \Sigma, \phi)$ and $Y \sim \mathcal{E}\mathcal{C}_d(\mu, \Sigma', \phi)$ be integrable with $\sigma_{ij} = \sigma'_{ij}$ for all $(i, j) \neq (1, 1)$ and $\sigma_{11} \leq \sigma'_{11}$. Then, $X \leq_{dcx} Y$.*

Proof. With a decomposition of Σ and Σ' that is similar to (3) with $\Sigma_{11} = (\sigma_{11})$ and $\Sigma'_{11} = (\sigma'_{11})$ one-dimensional it holds that

$$X_1 \,|\, (X_2, \ldots, X_d) = z \sim \mathcal{E}\mathcal{C}_2(\mu_z, \Sigma_{11.2}, \phi_{q(z)}), \quad Y_1 \,|\, (Y_2, \ldots, Y_d) = z \sim \mathcal{E}\mathcal{C}_2(\mu_z, \Sigma'_{11.2}, \phi_{q(z)}),$$

for $\Sigma_{11.2} = \sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, $\Sigma'_{11.2} = \sigma'_{11} - \Sigma'_{12}\Sigma'^{-1}_{22}\Sigma'_{21}$ and for some μ_z and $\phi_{q(z)}$ similar to (4). Since $\sigma_{11} \leq \sigma'_{11}$ it holds that $\Sigma_{11.2} \leq \Sigma'_{11.2}$ and, thus,

$$X_1 \,|\, (X_2, \ldots, X_d) = z \ \leq_{cx}\ Y_1 \,|\, (Y_2, \ldots, Y_d) = z$$

for all z, see Landsman and Tsanakas [21, Corollary 1]. Since the convex ordering and the directionally convex ordering coincide in the one-dimensional case, we obtain with the concatenation property of the directionally convex ordering that

$$X \,|\, (X_2, \ldots, X_d) = z \ \leq_{dcx}\ Y \,|\, (Y_2, \ldots, Y_d) = z$$

for all z. Then, the closure of the directionally convex ordering under mixtures implies $X \leq_{dcx} Y$ using that $(X_2, \ldots, X_d) \stackrel{\mathrm{d}}{=} (Y_2, \ldots, Y_d)$, see Müller and Stoyan [26, Theorem 3.12.6]. □

The following result provides ordering criteria for the directionally convex ordering in the class of elliptical distributions with a fixed generator, see also Yin [35, Theorem 3.6]. Our proof is based on Lemma 2 and on the characterization of the supermodular ordering in Theorem 1.

Theorem 2 ($\leq_{dcx}$-ordering of elliptical distributions for fixed generator)**.**
Let $X \sim \mathcal{EC}_d(\mu, \Sigma, \phi)$ and $Y \sim \mathcal{EC}_d(\mu', \Sigma', \phi)$ with $\Sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ and $\Sigma' = (\sigma'_{ij})_{1\leq i,j\leq d}$ be integrable. Then,

(i) *$\mu = \mu'$ and $\sigma_{ij} \leq \sigma'_{ij}$ f.a. i, j imply $X \leq_{dcx} Y$.*

(ii) *If $X \leq_{dcx} Y$, then $\mu = \mu'$. If additionally X and Y are square-integrable, then $\sigma_{ij} \leq \sigma'_{ij}$ f.a. i, j holds true.*

Proof. (i): Let $\xi \sim \mathcal{EC}_d(\mu, \Sigma'', \phi)$ where $\Sigma'' = (\sigma''_{ij})_{1\leq i,j\leq d}$ is given by $\sigma''_{ii} = \sigma'_{ii}$ for all i and $\sigma''_{ij} = \sigma_{ij}$ for all $i \neq j$. Since componentwise increasing the diagonal elements does not affect the positive semi-definiteness, Σ'' is positive semi-definite. Thus, Lemma 2 implies $X \leq_{dcx} \xi$. Due to Theorem 1, it holds that $\xi \leq_{sm} Y$ and, thus, $X \leq_{dcx} Y$.
(ii): Choosing the functions $f(x) = \pm x_i$ leads to $\mu = \mu'$. In the square-integrable case, $\sigma_{ij} \leq \sigma'_{ij}$ follows with $f(x) = (x_i - \mu_i)(x_j - \mu_j)$. □

Remark 2. It is so far not clear whether the assumption of square-integrability in Theorem 2(ii) can be omitted. This question also concerns related statements in the literature.

As a consequence of the above theorem, an increase of the radial variable w.r.t. the (univariate) stochastic order implies a directionally convex ordering result of the corresponding random vectors.

Corollary 1 ($\leq_{dcx}$-ordering of elliptical distributions for different generators)**.**
Let $X \sim \mathcal{EC}_d(\mu, \Sigma, \phi)$ and $Y \sim \mathcal{EC}_d(\mu, \Sigma', \phi')$ with $\Sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ and $\Sigma' = (\sigma'_{ij})_{1\leq i,j\leq d}$ be integrable. Assume that $\sigma_{ij} \leq \sigma'_{ij}$, $\sigma'_{ij} \geq 0$ f.a. i, j and $\phi \in \Phi_k$ for $k = \operatorname{rank}(\Sigma')$.
If the radial variables $R_{k,\phi}$ and $R_{k,\phi'}$ satisfy $R_{k,\phi} \leq_{st} R_{k,\phi'}$, then it follows that $X \leq_{dcx} Y$.

Proof. Consider $X' \sim \mathcal{EC}_d(\mu, \Sigma', \phi)$, which is well-defined due to the assumption that $\phi \in \Phi_k$, see (1) and (2). Then, Theorem 2 implies that $X \leq_{dcx} X'$. To show that $X' \leq_{dcx} Y$, let F and G be the distribution function of $R_{k,\phi}$ and $R_{k,\phi'}$, respectively. Then, $R_{k,\phi} \leq_{st} R_{k,\phi'}$ is equivalent to $F^{-1}(V) \leq G^{-1}(V)$ almost surely for some uniformly on $(0, 1)$ distributed random variable V, see, e.g., Shaked and Shantikumar [31, Theorem 1.A.1]. Consider the stochastic representations

$$X' = F^{-1}(V)U^{(k)}A \quad \text{and} \quad Y = G^{-1}(V)U^{(k)}A$$

as in (1) with $U^{(k)}$ independent of V. Then, the conditional distributions $(X' \mid V = v) \sim \mathcal{EC}_d(\mu, (F^{-1}(v))^2\Sigma', \Omega_k)$ and $(Y \mid V = v) \sim \mathcal{EC}_d(\mu, (G^{-1}(v))^2\Sigma', \Omega_k)$ are elliptically distributed with generator being the characteristic function Ω_k of the k-dimensional spherical distribution. Thus, we obtain from Theorem 2 that

$$(X' \mid V = v) \leq_{dcx} (Y \mid V = v)$$

for almost all v, using that all of $F^{-1}(v), G^{-1}(v), \sigma'_{ij}$ are non-negative. Hence, the statement follows from the closure of the directionally convex order under mixtures, see Müller [23]. □

3. Risk bounds in classes of elliptical models

As application of the ordering results in Section 2, we determine greatest elements w.r.t. $\leq_{sm}$ and $\leq_{dcx}$, respectively, in some classes of elliptical models. In risk applications, greatest elements, if they exist, correspond to worst case distributions in these models. Note that greatest elements w.r.t. $\leq_{sm}$ and $\leq_{dcx}$ are always unique because these orderings are partial orders and, thus, antisymmetric.

In typical applications in risk analysis, the risk vector is not completely specified but is given, e.g., by all elliptical models with certain bounds on the correlations or on the marginal distributions. As an important consequence of the supermodular and directionally convex ordering results, we obtain bounds of the aggregated risk for these models. Note that

$$(X_i)_i \leq_{sm} (Y_i)_i \quad \text{or} \quad (X_i)_i \leq_{dcx} (Y_i)_i \quad \Longrightarrow \quad \sum_i X_i \leq_{cx} \sum_i Y_i \quad \Longrightarrow \quad \Psi\Big(\sum_i X_i\Big) \leq \Psi\Big(\sum_i Y_i\Big)$$

for any law-invariant, convex, Fatou-continuous risk measure Ψ on an atomless probability space, see, e.g., Bäuerle and Müller [8, Theorem 4.3]. While the directionally convex ordering allows a comparison of random vectors with different marginal distributions, the (tighter) supermodular ordering requires equality of the marginal distributions, i.e., $X_i \stackrel{\mathrm{d}}{=} Y_i$ for all i. A useful property of the supermodular ordering is its invariance under increasing transformations of the components, i.e., if $(X_i)_i \leq_{sm} (Y_i)_i$ and $h_i \colon \mathbb{R} \to \mathbb{R}$ is increasing, $1 \leq i \leq d$, then $(h_i(X_i))_i \leq_{sm} (h_i(Y_i))_i$. Thus, an ordering result for X and Y implies ordering results w.r.t. transformed marginals. Note that the directionally convex ordering is invariant w.r.t. increasing convex transformations h_i of the components.

These invariance properties imply that the $\leq_{sm}$- and $\leq_{dcx}$-ordering results are applicable to models with an elliptical dependence structure and reasonable general marginals.

As first type of applications, we consider elliptical risk models $\mathcal{M}_1$ determined by upper bounds on the covariances of the form

$$\mathcal{M}_1 = \{X \sim \mathcal{EC}_d(\mu, \Sigma, \phi) \,|\, \Sigma \leq \Sigma^{\mathrm{u}} \text{ componentwise}\},$$

where $\Sigma^{\mathrm{u}} = (\sigma^{\mathrm{u}}_{ij}) \in \mathbb{R}^{d\times d}$ is a positive semi-definite, symmetric matrix and $\phi \in \Phi_d$ is an elliptical generator of the class Φ_d such that the corresponding radial variable R is integrable. It is thus assumed that the (generalized) covariances σ_{ij} of Σ are upper bounded by the covariances σ^{u}_{ij} of the matrix Σ^{u}.

As consequence of Theorem 2, we directly obtain an identification of a unique worst case distribution in model $\mathcal{M}_1$ w.r.t. the directionally convex order.

Corollary 2 ($\leq_{dcx}$**-maximum of** $\mathcal{M}_1$**).**
Let $Y \sim \mathcal{EC}_d(\mu, \Sigma^{\mathrm{u}}, \phi)$. Then, it holds that

(i) *$Y \in \mathcal{M}_1$, and*

(ii) *$X \leq_{dcx} Y$ for all $X \in \mathcal{M}_1$.*

As a second type of applications, we determine the worst case distribution w.r.t. the supermodular ordering for an elliptical model with given upper bounds on the (generalized) partial correlations corresponding to a canonical vine (C-vine) structure. Since the supermodular ordering is a pure dependence ordering and invariant under increasing transformations, our assumptions concern only the dependence structure and can also be applied to other marginal distributions. C-vine models are an important subclass of regular vine copula models which are a tool to model general dependencies, see, e.g., Kurowicka and Joe [20] and Czado [13].

For $\ell, \ell' \in \mathbb{N}_0$ with $\ell \leq \ell'$, denote by $\ell : \ell'$ the vector of indices $(\ell, \dots, \ell')$. If $\ell > \ell'$, set $\ell : \ell' = \emptyset$. Let $(\sigma_{ij,1:(i-1)})_{1\leq i<j\leq d} \in [-1,1]^{\frac{d(d-1)}{2}}$ be a given $\binom{d}{2}$-dimensional vector. For $j \in \{3, \dots, d\}$ and $i \in \{2, \dots, j-1\}$, define iteratively $\sigma_{ij,1:k}$ for $k \in \{i-1, \dots, 1\}$ by

$$\sigma_{ij,1:(k-1)} := \sigma_{ki,1:(k-1)}\sigma_{kj,1:(k-1)} + \sigma_{ij,1:k}\sqrt{1-\sigma^2_{ki,1:(k-1)}}\sqrt{1-\sigma^2_{kj,1:(k-1)}}\,. \tag{5}$$

Then, define the matrix $\Sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ by $\sigma_{ii} = 1$ for all i and by $\sigma_{ij} = \sigma_{ji} = \sigma_{ij,1:0}$ for $i < j$.

Let $\mathcal{M}^d_{\mathrm{cor}}$ be the set of correlation matrices, i.e., the set of positive semi-definite, symmetric $d \times d$ matrices with all diagonal elements equal to 1. By Proposition 1, Σ is a correlation matrix and, further, for any correlation matrix $\Sigma' = (\sigma'_{ij}) \in \mathcal{M}^d_{\mathrm{cor}}$, there exists a decomposition $(\sigma_{ij,1:(i-1)})_{1\leq i<j\leq d} \in [-1,1]^{\frac{d(d-1)}{2}}$ such that the recursive formula (5) leads to the matrix Σ', i.e., $\sigma_{ij} = \sigma'_{ij}$ for all i, j. If Σ is the correlation matrix of a square-integrable elliptically

distributed random vector $(Y_1, \dots, Y_d)$, then $\sigma_{ij,1:(i-1)}$ is the partial correlation of Y_i and Y_j given $Y_1, \dots, Y_{i-1}$ which coincides with the conditional correlation, see Baba et al. [7, Example 2].

For $\phi \in \Phi_d$ and $b_i \in [0,1]$, $1 \leq i < d$, consider the elliptical model

$$\mathcal{M}_2 := \{X \sim \mathcal{E}C_d(0, \Sigma, \phi) \,|\, \Sigma \in \mathcal{M}^d_{\mathrm{cor}} : |\sigma_{ij,1:(i-1)}| \leq b_i \text{ f.a. } i < j\} \tag{6}$$

with bounded (generalized) partial correlations corresponding to a canonical vine structure. Note that we do not pose any integrability assumption. Further, the univariate marginals are fixed and thus $\mathcal{M}_2$ is a pure dependence model. We aim to determine a greatest element of $\mathcal{M}_2$ in supermodular ordering. This also yields unique worst case distributions in models with transformed univariate marginal distributions using the invariance property of the supermodular ordering under increasing transformations.

The following result shows that the (generalized) partial correlations $(\sigma_{ij,1:(i-1)})_{1\leq i<j\leq d}$ are algebraically independent and determine a unique correlation matrix. More precisely, the set of positive definite correlation matrices can be characterized in terms of (generalized) partial correlations that correspond to a canonical vine (or C-vine) which is a star-shaped regular vine, see, e.g., Kurowicka and Cooke [18] and Aas et al. [1] for definitions.

Proposition 1.

(i) *There is a one-to-one correspondence between the set of $d \times d$ positive definite correlation matrices and the set of (generalized) partial correlations $(\sigma_{ij,1:(i-1)})_{1\leq i<j\leq d} \in (-1,1)^{\frac{d(d-1)}{2}}$ corresponding to a C-vine.*

(ii) *The (generalized) partial correlations $(\sigma_{ij,1:(i-1)})_{1\leq i<j\leq d} \in [-1,1]^{\frac{d(d-1)}{2}}$ determine a correlation matrix uniquely.*

(iii) *If $\Sigma \in \mathcal{M}^d_{cor}$ is not of full rank, the corresponding (generalized) partial correlations $(\sigma_{ij,1:(i-1)})_{1\leq i<j\leq d}$ are not necessarily uniquely determined.*

Proof. (i): The (generalized) partial correlations correspond to the structure of a canonical vine. Thus, the statement follows from Bedford and Cooke [9, Corollary 7.5].
Statement (ii) is a consequence of (i) and (5).
(iii): The determinant of Σ is given by

$$\det(\Sigma) = \prod_{i=1}^{d-1} \prod_{j=i+1}^{d} (1 - \sigma^2_{ij,1:(i-1)}),$$

see Kurowicka and Cooke [19, Theorem 4.5]. Thus, the determinant vanishes if and only if there exist $1 \leq i < j \leq d$ such that $\sigma_{ij,1:(i-1)} \in \{-1, 1\}$. In this case, (5) implies that the (generalized) partial correlations $\sigma_{\ell j,1:(\ell-1)}$, $i < \ell < j$, are not uniquely determined. $\square$

For $\Sigma = (\sigma_{ij})$ and $\Sigma' = (\sigma'_{ij})$, the following proposition gives some elementary ordering results w.r.t. the (generalized) partial correlations based on formula (5). To keep the notation simple, we formulate it in the case that $k = 1$, $i = 2$ and $j = 3$.

Proposition 2 (Ordering partial correlations)**.**
For the (generalized) partial correlations the following ordering properties hold true:

(i) *If $\sigma_{1\ell} = \sigma'_{1\ell}$ for $\ell \in \{2,3\}$, then $\sigma_{23,1} \leq \sigma'_{23,1}$ implies $\sigma_{23} \leq \sigma'_{23}$.*

(ii) *If $\sigma_{23,1} = \sigma'_{23,1}$, then $0 \leq |\sigma_{1\ell}| \leq \sigma'_{12} = \sigma'_{13}$ for $\ell \in \{2,3\}$ implies $\sigma_{23} \leq \sigma'_{23}$.*

(iii) *If $\sigma_{23,1} = \sigma'_{23,1} \leq 0$, then $0 \leq \sigma_{1\ell} \leq \sigma'_{1\ell}$ for $\ell \in \{2,3\}$ implies $\sigma_{23} \leq \sigma'_{23}$.*

Proof. Statement (i) follows from the partial correlation formula

$$\sigma_{23} = \sigma_{12}\sigma_{13} + \sigma_{23,1}\sqrt{1-\sigma_{12}^2}\sqrt{1-\sigma_{13}^2} =: f(\sigma_{23,1}, \sigma_{12}, \sigma_{13}), \tag{7}$$

see (5). The partial derivative $\partial_2 f = \frac{\partial f}{\partial \sigma_{12}}$ of f w.r.t. the second variable is given by

$$\partial_2 f(\sigma_{23,1}, \sigma_{12}, \sigma_{13}) = \sigma_{13} - \frac{\sigma_{23,1}\sqrt{1-\sigma_{13}^2}\,\sigma_{12}}{\sqrt{1-\sigma_{12}^2}}. \tag{8}$$

Then, statement (ii) follows from

$$f(a, c, d) \le f(a, d, d) \le f(a, e, e)$$

f.a. $a \in [-1, 1]$, and $0 \le |c| \le d \le e \le 1$ where the first inequality holds true because

$$|ac| \le d \quad \Longrightarrow \quad ac\sqrt{1-d^2} \le d\sqrt{1-c^2} \quad \Longrightarrow \quad \partial_2 f(a, c, d) \ge 0\,.$$

The second inequality is fulfilled since for $g_a(s) := f(a, s, s)$ holds $g_a'(s) = 2(1-a)s \ge 0$ f.a. $a \le 1$ and $s \ge 0$. Statement (iii) is a consequence of (8). □

For the bounds $(b_i)_i$ on the (generalized) partial correlations in model $\mathcal{M}_2$, define the numbers $a_1, \dots, a_{d-1} \in [0, 1]$ iteratively by

$$\begin{aligned} a_{i,i-1} &:= b_i\,, && i \in \{1, \dots, d-1\}\,, \\ a_{i,k-1} &:= a_{k,k-1}^2 + a_{i,k}(1 - a_{k,k-1}^2)\,, && k \in \{i-1, \dots, 1\}\,, \\ a_i &:= a_{i,0}\,, && i \in \{1, \dots, d-1\}\,. \end{aligned} \tag{9}$$

Denote by δ_{ij} the Kronecker delta and by $i \wedge j$ the minimum of i and j. According to the following result, there exists a (unique) worst case distribution in model $\mathcal{M}_2$ w.r.t. the supermodular ordering which is given as follows.

Theorem 3 (Bounded partial correlations)**.**
Let $Y \sim \mathcal{E}\mathcal{C}_d(0, \Sigma^{\mathrm{u}}, \phi)$ be elliptically distributed with $\Sigma^{\mathrm{u}} = (\sigma_{ij}^{\mathrm{u}})_{1\le i,j\le d}$ given by $\sigma_{ii}^{\mathrm{u}} = 1$ for all i and $\sigma_{ij}^{\mathrm{u}} = a_{i\wedge j}$ for all $i \neq j$. Then, it holds that

(i) *$Y \in \mathcal{M}_2$, and*

(ii) *$X \le_{sm} Y$ for all $X \in \mathcal{M}_2$.*

Proof. Applying the partial correlation formula (5) for $2 \le i < j \le d$ inductively over $k \in \{i-1, \dots, 1\}$ yields to

$$\begin{aligned} \sigma_{ij,1:(k-1)} &= \sigma_{ki,1:(k-1)}\,\sigma_{kj,1:(k-1)} + \sigma_{ij,1:k}\sqrt{1-\sigma_{ki,1:(k-1)}^2}\sqrt{1-\sigma_{kj,1:(k-1)}^2} \\ &\le \sigma_{ki,1:(k-1)}\,\sigma_{kj,1:(k-1)} + a_{i,k}\sqrt{1-\sigma_{ki,1:(k-1)}^2}\sqrt{1-\sigma_{kj,1:(k-1)}^2} \le a_{k,k-1}^2 + a_{i,k}\cdot(1 - a_{k,k-1}^2) = a_{i,k-1} \end{aligned}$$

using Proposition 2 (i), (ii) and (9). This implies with $\sigma_{1,j} \le b_1 = a_1$ for $j \in \{2, \dots, d\}$ that $\sigma_{ij} \le a_i$ for all $1 \le i < j \le d$. Since $\sigma_{ij} = \sigma_{ji}$ for all $i \neq j$, it follows that $\sigma_{ij} \le a_{i\wedge j}$ for all $i \neq j$. Choosing $(\sigma_{ij,1:(i-1)})_{1\le i<j\le d} = b_i$ leads to $\sigma_{ij} = a_i$ for all $1 \le i < j \le d$. This defines a correlation matrix (see Proposition 1 (ii)) which coincides with Σ^{u}. □

Remark 3.

(a) If $b_1 = 1$ in (6), then by construction $a_i = 1$ for all $1 \leq i < d$. This leads to $\sigma^{\mathrm{u}}_{ij} = 1$ f.a. $1 \leq i, j \leq d$, and, hence, $Y \overset{\mathrm{d}}{=} (X_1^c, \ldots, X_d^c)$ is the standard comonotonic upper bound for $X = (X_1, \ldots, X_d)$ w.r.t. the supermodular ordering, i.e., there is no improvement of the standard bounds. This coincides with the fact that $a_1 = 1$ yields $\sigma^{\mathrm{u}}_{1i} = 1$ (which means $\mathrm{Cor}(Y_1, Y_i) = 1$ in the square-integrable case) f.a. i, and thus $Y = (Y_1, \ldots, Y_d)$ is comonotonic. In this case, the (generalized) correlations $(\sigma_{1j})_{1\leq j\leq d}$ determine the correlation matrix uniquely and the (generalized) partial correlations $(\sigma_{ij,1:i})_{2\leq i<j\leq d}$ are not uniquely determined, see Proposition 1(iii).

More generally, if $b_i = 1$ for some $i \in \{1, \ldots, d-1\}$ in (6) then $a_j = 1$ for all $i \leq j < d$. This implies that $(Y_i, \ldots, Y_d)$ is comonotonic and $(Y \mid Y_{1:(i-1)} = y)$ is comonotonic conditionally on $y \in \mathbb{R}^{i-1}$. The case $b_2 = 1$ is a special case of Ansari and Rüschendorf [6, Theorem 2.3] in the context of partially specified internal risk factor models.

(b) If $b_i = 0$ for all $1 \leq i < d$, then $a_i = 0$ for all i and thus $\Sigma = \Sigma^{\mathrm{u}} = (\delta_{ij})_{1\leq i,j\leq d}$, i.e. $X \overset{\mathrm{d}}{=} Y$ has uncorrelated components. Note that the components are only independent in the case of a multivariate normal distribution, see Cambanis et al. [12, Section 5(d)].

(c) The bounds $(b_i)_i$ in (6) on the (generalized) (partial) correlations $\sigma_{ij|1:(i-1)}$ lead to a positive semi-definite matrix Σ^{u} defined in Theorem 3 which implies improved risk bounds. Note that, in general, upper bounds on the unconditional (generalized) correlations $(\sigma_{ij})_{ij}$ do not specify a positive semi-definite matrix and, thus, it is not clear whether a worst case distribution exists and how to obtain good bounds in this case.

The following example illustrates Theorem 3.

Example 1. Assume that $X \sim \mathcal{EC}_4(0, \Sigma, \phi)$ where $\Sigma = (\sigma_{ij})_{1\leq i,j\leq 4} \in \mathcal{M}^4_{\mathrm{cor}}$ with (generalized) partial correlations corresponding to the C-vine in Fig. 1(a). Assume that

$$|\sigma_{12}|, |\sigma_{13}|, |\sigma_{14}| \leq 0.5 = b_1 = a_{1,0}\,, \quad |\sigma_{23,1}|, |\sigma_{24,1}| \leq 0.6 = b_2 = a_{2,1}\,, \quad |\sigma_{34,12}| \leq 0.4 = b_3 = a_{3,2}\,. \tag{10}$$

Then, Theorem 3 leads to

$$a_1 = a_{1,0} = 0.5\,, \qquad a_2 = a_{2,0} = a_{1,0}^2 + a_{2,1}(1 - a_{1,0}^2) = 0.7\,,$$
$$a_{3,1} = a_{2,1}^2 + a_{3,2}(1 - a_{2,1}^2) = 0.616\,, \qquad a_3 = a_{3,0} = a_{1,0}^2 + a_{3,1}(1 - a_{1,0}^2) = 0.712\,.$$

Hence, for $Y \sim \mathcal{EC}_4(0, \Sigma^{\mathrm{u}}, \phi)$ with

$$\Sigma^{\mathrm{u}} = \begin{pmatrix} 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1 & 0.7 & 0.7 \\ 0.5 & 0.7 & 1 & 0.712 \\ 0.5 & 0.7 & 0.712 & 1 \end{pmatrix},$$

holds $X \leq_{sm} Y$, i.e., $\mathcal{EC}_4(0, \Sigma^{\mathrm{u}}, \phi)$ is the unique worst case distribution in the class of elliptical models with bounds on the partial correlations as in (10).

In the previous example, we have illustrated Theorem 3 which shows that a (componentwise) greatest element in the class of correlation matrices with bounded partial correlations corresponding to a C-vine structure exists and can be determined by the recursive formula (5). In the following example, we show that this result cannot be generalized to regular vine structures. A componentwise greatest element in the class of correlation matrices with bounded partial correlations corresponding to a regular vine that is not a C-vine does not necessarily exist. Since every regular vine that is not a C-vine contains a sub-vine that is a D-vine on 4 variables, we consider without loss of generality a D-vine structure on 4 variables.

Example 2. Consider the D-vine in Fig. 1(b). Then, a correlation matrix $\Sigma = (\sigma_{ij})_{1\leq i\leq j\leq 4} \in \mathcal{M}^d_{\mathrm{cor}}$ is uniquely determined by the (partial) correlations $\sigma_{12}, \sigma_{23}, \sigma_{34}, \sigma_{13,2}, \sigma_{24,3}, \sigma_{14,23} \in [-1, 1]$, see Bedford and Cooke [9, Corollary

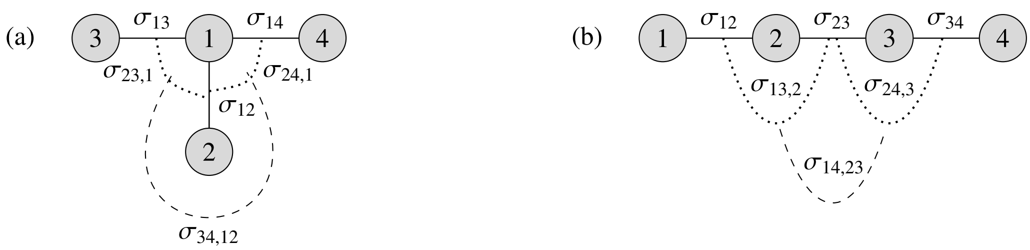

Fig. 1. A complete partial correlation C-vine specification (a) and a complete partial correlation D-vine specification (b) on 4 variables.

7.5]. From Cambanis et al. [12, Corollary 5], we obtain by some elementary but tedious calculations for $\sigma_{23} \neq \pm 1$ that

$$\begin{aligned}\sigma_{14} = &\frac{1}{1-\sigma_{23}^2}\left(\sigma_{12}\sigma_{24} + \sigma_{13}\sigma_{34} - \sigma_{13}\sigma_{23}\sigma_{24} - \sigma_{12}\sigma_{23}\sigma_{34}\right) \\ &+ \frac{1}{1-\sigma_{23}^2}\sigma_{14,23}\sqrt{1-\sigma_{23}^2-\sigma_{12}^2-\sigma_{13}^2+2\sigma_{12}\sigma_{13}\sigma_{23}}\sqrt{1-\sigma_{23}^2-\sigma_{24}^2-\sigma_{34}^2+2\sigma_{23}\sigma_{24}\sigma_{34}}\,,\end{aligned} \tag{11}$$

where

$$\begin{aligned}\sigma_{13} &= \sigma_{12}\sigma_{23} + \sigma_{13,2}\sqrt{1-\sigma_{12}^2}\sqrt{1-\sigma_{23}^2}\,, \\ \sigma_{24} &= \sigma_{23}\sigma_{34} + \sigma_{24,3}\sqrt{1-\sigma_{23}^2}\sqrt{1-\sigma_{34}^2}\,.\end{aligned} \tag{12}$$

For the class of bounded partial correlations $(\sigma_{ij,(i+1):(j-1)})_{1\leq i<j\leq 4}$ corresponding to a D-vine structure such that

$$|\sigma_{ij,(i+1):(j-1)}| \leq b_{j-i}\,, \quad 1 \leq i < j \leq 4\,, \tag{13}$$

for $b_{j-i} \in [0,1]$, we show that σ_{14} is not necessarily maximum if it is a function of the bounds b_{j-i}.

Consider for $b_1 = \frac{\sqrt{2}}{2}$, $b_2 = \frac{1}{2}$, and $b_3 = 1$ the class

$$\mathcal{S} = \left\{\Sigma = (\sigma_{ij})_{1\leq i,j\leq 4} \in \mathcal{M}^4_{\mathrm{cor}} \colon |\sigma_{ij,(i+1):(j-1)}| \leq b_{j-i}\,, 1 \leq i < j \leq 4\right\} \tag{14}$$

of correlation matrices defined by the partial correlations $(\sigma_{ij,(i+1):(j-1)})_{1\leq i<j\leq 4}$ which are bounded by $\pm b_{j-i}$. Choose the matrices $\Sigma' = (\sigma'_{ij}), \Sigma^* = (\sigma^*_{ij}) \in \mathcal{S}$ defined by

$$\begin{aligned}\sigma'_{12} = \sigma'_{34} = \sigma^*_{12} = \sigma^*_{23} = \sigma^*_{34} &= b_1\,, \quad \sigma'_{23} = \frac{1}{2}\,, \\ \sigma'_{13,2} = \sigma'_{24,3} = \sigma^*_{13,2} = \sigma^*_{24,3} &= b_2\,, \\ \sigma'_{14,23} = \sigma^*_{14,23} &= b_3\,.\end{aligned}$$

Then, Σ^* only depends on the (partial) correlation bounds b_i, $1 \leq i \leq 3$. It holds that

$$\begin{aligned}\sigma'_{13} = \sigma'_{24} &= \frac{1}{4}\sqrt{2} + \frac{1}{8}\sqrt{6} \approx 0.6597396\,, & \sigma^*_{13} = \sigma^*_{24} &= \frac{3}{4}\,, \\ \sigma'_{14} &= \frac{9}{16} + \frac{1}{4}\sqrt{3} \approx 0.9955127\,, & \sigma^*_{14} &= \frac{3}{8} + \frac{7}{16}\sqrt{2} \approx 0.9937184\,,\end{aligned}$$

and, thus, $\sigma'_{13} < \sigma^*_{13}$ and $\sigma'_{24} < \sigma^*_{24}$, but $\sigma'_{14} > \sigma^*_{14}$. Hence, contrary to the C-vine case in (6) and (10), a componentwise greatest element in the class $\mathcal{S}$ of correlation matrices with bounded partial correlations corresponding to a D-vine structure does not exist. While the correlations σ_{13} and σ_{24} in (12) are maximum given the constraints (13) if and only if $\sigma_{12} = \sigma_{23} = \sigma_{34} = b_1$ and $\sigma_{13,2} = \sigma_{24,3} = b_2$ (see Proposition 2), the correlation σ_{14} in (11) is not maximum for the maximal choice in (13), i.e., for $\sigma_{12} = \sigma_{23} = \sigma_{34} = b_1$, $\sigma_{13,2} = \sigma_{24,3} = b_2$, and $\sigma_{14,23} = b_3$, even though, by (11), σ_{14} is monotone in $\sigma_{14,23}$. As a consequence, a greatest element in $\mathcal{S}$ does not exist.

For the set of correlation matrices with bounded partial correlations specifying a C-vine structure (like in Theorem 3), a unique correlation matrix, such that

$$\mathbb{E}f(X)\,, \quad X \sim \mathcal{E}C_d(\mu, \Sigma, \phi)\,, \tag{15}$$

is maximized over all such correlation matrices simultaneously for all supermodular functions f, exists and can be determined. In the above example, we have shown that this is, in general, not possible for non-C-vine structures. However, for a fixed supermodular function f, a maximization of (15) w.r.t. such a constrained set of correlation matrices reduces to a convex optimization problem as follows.

Remark 4. By an argument in Giovagnoli and Romanazzi [17, Lemma 3 and proof of Proposition 3], the map of a correlation matrix R to the matrix R^* of partial correlations is matrix concave. As a consequence of the quasi-convexity of the absolute value $|\tau|$ of the correlation τ, therefore, the set $\mathcal{S}$ in (14) is convex. The maximal elements of $\mathcal{S}$ are determined by maximizing $\mathbb{E}f(X)$ over $X \in \mathcal{E}C_d(\mu, \Sigma, \phi)$, $\Sigma \in \mathcal{S}$, which due to the convexity of $\mathcal{S}$ can be characterized as a solution of a dual problem. Alternatively, this maximization problem can be solved numerically by maximizing over a bounded set of parameters. This remark also extends to general regular vine structures.

4. Worst case distributions in partially specified factor models

The ordering results in Section 3 make use of the dependence structure of a full elliptical model which may be not easy to justify in applications. This is the reason for the introduction of partially specified factor models (PSFM) (X, Z) given by a risk vector $X = (X_1, \dots, X_d)^\top$ and a real-valued risk factor Z such that $X_i = f_i(Z, \varepsilon_i)$, $1 \leq i \leq d$, where $(\varepsilon_1, \dots, \varepsilon_d)$ are idiosyncratic risks independent of Z. The dependence structure among the $(\varepsilon_i)_i$ is not specified in contrast to the usual independence assumption in factor models. Here, only the distribution of (X_i, Z) is specified, $1 \leq i \leq d$. This simplified assumption makes the PSFM a very flexible and general type of models compared to the fully specified models.

Partially specified risk factor models (PSFM's) are of considerable practical relevance for the reduction of (upper) risk bounds because they improve the comonotonic upper bound w.r.t. $\leq_{sm}$ obtained for marginal models where only the univariate marginal distributions are specified (see Bernard et al. [10]). In PSFM's, the upper bound w.r.t. $\leq_{sm}$ is given by a conditionally comonotonic vector. Similar results for lower bounds and, in particular, minimum elements w.r.t. $\leq_{sm}$ are in general only available for the bivariate case.

In the following, we only assume that the dependence of (X_i, Z) is elliptical. In the first part of this section, we analyze risk bounds w.r.t. the supermodular ordering where the dependence structure of (X_i, Z) is fixed. In the second part of this section, we derive directionally convex ordering results where we allow the copula of (X_i, Z) to come from some sub-families of elliptical distributions and the marginal distributions to come from some sets of distribution functions with upper bounds in convex order.

4.1. Bounds w.r.t. the supermodular ordering

For $\Sigma = \left(\begin{smallmatrix} 1 & \rho \\ \rho & 1 \end{smallmatrix}\right)$, $\rho \in [-1, 1]$, we abbreviate the bivariate distribution $\mathcal{E}C_2(0, \Sigma, \phi)$ by $\mathcal{E}C_2(0, \rho, \phi)$. For $\phi \in \Phi_2$ and for (generalized) correlations $\rho_i \in [-1, 1]$, $1 \leq i \leq d$, consider the class of PSFM's

$$\mathcal{M}_3^f := \{(X, Z) \,|\, (X_i, Z) \sim \mathcal{E}C_2(0, \rho_i, \phi)\,, 1 \leq i \leq d\} \tag{16}$$

with bivariate elliptical specifications of each component X_i and the common risk factor $Z \sim \mathcal{E}C_1(0, 1, \phi)$.

We derive $\leq_{sm}$-ordering results for the upper bounds (w.r.t. $\leq_{sm}$) of the risk vector class

$$\mathcal{M}_3 := \{X \,|\, \exists Z \text{ such that } (X, Z) \in \mathcal{M}_3^f\}\,,$$

i.e., $\mathcal{M}_3$ is the projection of $\mathcal{M}_3^f$ on the set of risk vectors. Note that the assumptions in model $\mathcal{M}_3$ are quite mild. They in fact only concern the specification of fixed marginal distributions and the copula of (X_i, Z). Bounds for $\mathcal{M}_3$ directly lead by the transformation invariance to bounds for the class of PSFM's

$$\mathcal{M}_3'^f := \left\{(Y_1, \dots, Y_d, Z') \,|\, \exists (X, Z) \in \mathcal{M}_3^f \text{ with } Y_i = h_i(X_i)\,,\ Z' = g(Z)\right\}$$

for increasing functions h_i and a strictly increasing function g , and to bounds for the corresponding risk vector class

$$\mathcal{M}_3' = \{Y = (Y_1, \dots, Y_d) \,|\, \exists Z \text{ such that } (Y, Z) \in \mathcal{M}_3'^f\}\,.$$

As consequence, ordering results for $\mathcal{M}_3$ directly imply corresponding ordering results for the class $\mathcal{M}_3'$ with general marginals.

Define $M\colon [-1,1]^2 \to [-1,1]$ by

$$M(a,b) := ab + \sqrt{1-a^2}\,\sqrt{1-b^2}\,.$$

Let $X_{i,z}^c := F^{-1}_{X_i|Z=z}(U)$ with $U \sim U(0,1)$ independent of Z. Then, the conditionally comonotonic random vector $X_Z^c = (X_{1,Z}^c, \dots, X_{d,Z}^c)$ has the (uniquely determined) worst case distribution in the PSFM $\mathcal{M}_3$ in (16) w.r.t. the supermodular ordering as follows, see Ansari and Rüschendorf [4, Theorem 2].

Proposition 3. *For the conditionally comonotonic vector X_Z^c it holds that*

(i) $X_Z^c \in \mathcal{M}_3$ *and*

(ii) $X \leq_{sm} X_Z^c$ *for all* $X \in \mathcal{M}_3$.

Further, $(X_Z^c) \sim \mathcal{E}C_d(0, \Sigma, \phi)$ *is elliptically distributed where* $\Sigma = (\sigma_{ij})$ *is given by*

$$\sigma_{ij} = \begin{cases} 1 & \text{for } i = j\,, \\ M(\rho_i, \rho_j) & \text{for } i \neq j\,. \end{cases}$$

As a consequence of Theorem 1, we obtain the following result which characterizes the supermodular comparison of the worst case scenarios in models $\mathcal{M}_3 = \mathcal{M}_3((\rho_i)_i)$ w.r.t. the elliptical specifications $(\rho_i)_i$. It strengthens the lower orthant ordering result in classes of elliptical distributions in Ansari and Rüschendorf [4, Proposition 4] to the supermodular ordering.

Theorem 4. *Let* $(X_i, Z) \sim \mathcal{E}C_2(0, \rho_i, \phi)$, $(Y_i, Z) \sim \mathcal{E}C_2(0, \rho_i', \phi)$, $1 \leq i \leq d$. *Then, for conditionally comonotonic random vectors* X_Z^c *and* Y_Z^c *with these specifications holds*

$$X_Z^c \leq_{sm} Y_Z^c \quad \Longleftrightarrow \quad M(\rho_i, \rho_j) \leq M(\rho_i', \rho_j')\,, \ \forall i \neq j\,. \tag{17}$$

Remark 5.

(a) It can easily be verified that $M(a,b) = 1$ if and only if $a = b$. Thus, $\rho_i = \rho_j$ for all $i \neq j$ yields $X_Z^c \overset{\mathrm{d}}{=} X^c$, where $X^c = (F_{X_i}^{-1}(U))_{1 \leq i \leq d}$ is comonotonic. Thus, in all other cases the risk bounds in the PSFM improve on the pure marginal model.

(b) A sufficient condition for the ordering on the right hand side in (17) is

$$\rho_1 \gtrless \rho_1' \gtrless \rho_2 \gtrless \cdots \gtrless \rho_d \quad \text{and} \quad \rho_i' = \rho_i \text{ for all } 2 \leq i \leq d\,.$$

This is a special case of the sign-change ordering condition for upper products of bivariate copulas in the elliptical setting, see Ansari and Rüschendorf [5, Corollary 3.11]. In particular, also

$$\begin{gathered} \rho_1 \gtrless \cdots \gtrless \rho_k \gtrless \rho_k' \gtrless \rho_{k+1}' \gtrless \rho_{k+1} \gtrless \cdots \gtrless \rho_d\,, \\ \rho_1' = \cdots = \rho_k' \gtrless \rho_{k+1}' = \cdots = \rho_d \end{gathered} \tag{18}$$

for some $k \in \{1, \dots, d-1\}$ implies the ordering on the right hand side in (17).

(c) For $d = 2$, a sharp lower bound of $\mathcal{M}_3$ w.r.t. the supermodular ordering is given by the conditionally countermonotonic vector

$$(F^{-1}_{X_1|Z}(U), F^{-1}_{X_2|Z}(1-U))\,.$$

For an ordering criterion on lower bounds similar to Theorem 4, see Ansari and Rüschendorf [5, Remark 6]. As a slight extension, for $d = 3$ and $\rho_1 = 1$, a sharp lower bound of $\mathcal{M}_3$ w.r.t. the supermodular ordering is given by $(Z, F^{-1}_{X_1|Z}(U), F^{-1}_{X_2|Z}(1-U))$.

(d) If $\phi \in \Phi_k$ for some $k \geq 2$ and if $G_k(0) = 0$ for G_k defined in (2), then $X \sim \mathcal{E}C_d(\mu, \Sigma, \phi)$ has continuous univariate marginal distribution functions independent of μ and Σ . This is a consequence of the absolute continuity on $(0, \infty)$ of the radial variable corresponding to the univariate marginals of X , see Cambanis et al. [12, Corollary 2]. In this case, solutions of $\mathcal{M}_3'$ can be obtained and ordered for any fixed marginal distributions.

4.2. Bounds w.r.t. the directionally convex ordering

In the following, we extend the setting of the PSFM $\mathcal{M}_3$ in (16) to partial specification sets and to the consideration of different elliptical generators and different marginal distributions. More precisely, we consider an elliptical PSFM where the partial specifications are allowed to come from elliptical distributions with different generators and with a bound on the correlations. Further, we consider a PSFM where the univariate marginals are allowed to come from some large classes of distributions with upper bounds in convex order.

Let $-1 \leq \rho_1 < \rho_2 \leq 1$ such that $M(\rho_1, \rho_2) \geq 0$. For $p \in \{1, \ldots, d-1\}$ and $b_i > 0$, $1 \leq i \leq d+1$, consider the set

$$\mathcal{S}^{\rho_1,\rho_2} = \left\{\Sigma = (\sigma_{ij}) \in \mathcal{M}^{d+1}_{\mathrm{cor}} \,|\, \sigma_{i,d+1} \leq \rho_1 < \rho_2 \leq \sigma_{j,d+1} \text{ for } 1 \leq i \leq p < j \leq d\right\}$$

of correlation matrices with a constraint on the (generalized) correlation between the i-th component and the $(d+1)$-st by an upper bound ρ_1 if $i \leq p$, and by a lower bound ρ_2 if $i > p$.

For a mean vector $\mu \in \mathbb{R}^{d+1}$ and an elliptical generator $\phi \in \Phi_2$, consider the class of PSFM's

$$\mathcal{M}_4^f = \left\{(X, Z) \sim \mathcal{E}C_{d+1}(\mu, \Sigma, \psi) \,|\, \Sigma \in \mathcal{S}^{\rho_1,\rho_2} , \psi \in \Phi_{\mathrm{rank}(\Sigma)} , R_{2,\psi} \leq_{st} R_{2,\phi}\right\}$$

of elliptical distributions with partial dependence specifications of (X_i, Z) given by $\Sigma \in \mathcal{S}^{\rho_1,\rho_2}$ and with generator ψ whose radial variable $R_{2,\psi}$ is upper bounded w.r.t. the stochastic order by the radial variable $R_{2,\phi}$ corresponding to ϕ . Note that the risk factor Z is a real-valued random variable.

Again, the corresponding risk vector class is denoted by

$$\mathcal{M}_4 = \left\{X \,|\, \exists Z \text{ such that } (X, Z) \in \mathcal{M}_4^f\right\} . \tag{19}$$

In dependence on the choice of ϕ , this class comprises, e.g., multivariate normal and multivariate t-distributions with upper bounded, respectively, lower bounded correlations of (X_i, Z) . A worst case distribution of $\mathcal{M}_4$ w.r.t. the directionally convex ordering immediately yields to worst case distributions of models with increasing convex transformed marginal distributions.

To allow are more flexible modeling of the univariate marginal distributions, we also consider the following model. Let $\eta \in \Phi_2$ be the generator of a bivariate elliptical distribution that fulfills the positive dependence condition (21) with $\rho = M(\rho_1, \rho_2)$. Consider the families

$$\begin{aligned} C^{\rho_1,\eta} &= \{C \in C_2 \,|\, C \text{ is a copula of } \mathcal{E}C_2(0, r, \eta) , r \leq \rho_1\} , \\ C^{\rho_2,\eta} &= \{C \in C_2 \,|\, C \text{ is a copula of } \mathcal{E}C_2(0, r, \eta) , r \geq \rho_2\} \end{aligned}$$

of bivariate elliptical copulas with generator η and a correlation that is upper bounded by ρ_1 , respectively, lower bounded by ρ_2 .

For fixed distribution functions $F_i \in \mathcal{F}^1$, define the sets

$$\mathcal{F}_i := \{F \,|\, F \leq_{cx} F_i\}$$

of marginal distribution functions that are upper bounded by F_i in convex order, $1 \leq i \leq d$. Consider the class of PSFM's

$$\mathcal{M}_5^f = \left\{(X,Z) \,|\, F_{X_i} \in \mathcal{F}_i \,, C_{X_\ell,Z} \in C^{\rho_1,\eta} \,, C_{X_j,Z} \in C^{\rho_2,\eta} \,, 1 \leq \ell \leq p < j \leq d\right\}$$

and the related risk vector class

$$\mathcal{M}_5 = \left\{X \,|\, \exists Z \text{ such that } (X,Z) \in \mathcal{M}_5^f\right\} \tag{20}$$

with marginal specification sets $\mathcal{F}_i$ and elliptical dependence specification sets $C^{\rho_1,\eta}$ and $C^{\rho_2,\eta}$.

We aim to determine a greatest element of $\mathcal{M}_4$ and $\mathcal{M}_5$, respectively, w.r.t. the directionally convex order. Note that a greatest w.r.t. the supermodular ordering cannot be obtained because the elements in these classes do not all have identical univariate marginal distribution.

To obtain a greatest element of $\mathcal{M}_5$ w.r.t. $\leq_{dcx}$, we need the following positive dependence notion. Let $\xi = (\xi_1, \ldots, \xi_d)$ be a d-dimensional random vector. Then ξ is said to be conditionally increasing (CI) if for all $i \in \{1, \ldots, d\}$, $\xi_i \uparrow_{st} \xi_J$ for all $J \subset \{1, \ldots, d\} \setminus \{i\}$, i.e., $\mathbb{E}[f(\xi_i) \,|\, \xi_j = x_j \,, j \in J]$ is increasing in x_j for all $j \in J$, $J \subset \{1, \ldots, d\} \setminus \{i\}$ and for all non-decreasing functions f for which the expectation exists.

For a bivariate elliptical vector $\zeta \sim \mathcal{EC}_2(0, \rho, \psi)$, $\rho \in (-1, 1)$, with absolute continuous radial variable $R_{2,\psi}$, the density of ζ is of the form

$$h(z) = \frac{1}{\sqrt{1-\rho^2}} g\left(z \begin{pmatrix} \frac{1}{1-\rho^2} & \frac{\rho}{1-\rho^2} \\ \frac{\rho}{1-\rho^2} & \frac{1}{1-\rho^2} \end{pmatrix} z^\top\right), \quad z \in \mathbb{R}^2,$$

where g is a scale function that is uniquely determined by the distribution of $R_{2,\psi}$. A sufficient CI-condition for bivariate elliptical distributions is given as follows.

Lemma 3 (CI-criterion for bivariate elliptical distributions)**.**
Let $X \sim \mathcal{EC}_2(\mu, \rho, \eta)$, $\rho \in [0, 1)$, be a bivariate elliptical random vector with a scale function g such that $\beta(t) = \log(g(t))$ is twice differentiable. If

$$\beta''(t) = 0 \quad \textit{whenever} \quad \beta'(t) = 0 \quad \textit{and if} \quad -\frac{\rho}{1+\rho} \leq \inf_{t \in T} \frac{t\beta''(t)}{\beta'(t)} \leq \sup_{t \in T} \frac{t\beta''(t)}{\beta'(t)} \leq \frac{\rho}{1-\rho}, \tag{21}$$

where $T = \{t \in \mathbb{R}_+ : \beta'(t) < 0\}$, then X is conditionally increasing.

Proof. The statement follows from the characterization of likelihood ration dependence in Abdous et al. [2, Proposition 1.2]. □

Remark 6. Condition (21) is fulfilled for $\rho = 0$ only in the case that X is normally distributed. More generally, if $\rho = 0$ and $X \sim \mathcal{EC}_2(\mu, \rho, \eta)$ is CI, then X is normally distributed, see Abdous et al. [2, Proposition 1.3]. For normal random vectors, CI is characterized by the property that the inverse correlation function is an M-matrix (see Rüschendorf [27]). The above stated result implies that an extension of this characterization to elliptical distributions given in Tibiletti [33, Lemma 4.1] and Rüschendorf and Witting [29, Proposition 3.3] does not hold.

For the comparison of conditionally comonotonic elliptical risk vectors in the models $\mathcal{M}_4$ and $\mathcal{M}_5$, define the matrix $\Sigma = (\sigma_{ij})_{1 \leq i,j \leq d+1}$ by

$$\sigma_{ij} = \begin{cases} 1 & \text{if } 1 \leq i,j \leq p \text{ or } p < i,j \leq d \text{ or } i = j = d+1\,, \\ M(\rho_1, \rho_2) & \text{if } 1 \leq i \leq p < j \leq d \text{ or } 1 \leq j \leq p < i \leq d\,, \\ \rho_1 & \text{if } 1 \leq i \leq p\,, j = d+1\,, \text{ or } 1 \leq j \leq p\,, i = d+1\,, \\ \rho_2 & \text{if } p < i \leq d\,, j = d+1\,, \text{ or } p < j \leq d\,, i = d+1\,. \end{cases} \tag{22}$$

Then, the (uniquely determined) worst case distribution for the model $\mathcal{M}_4$ in (19), respectively, $\mathcal{M}_5$ in (20) w.r.t. the directionally convex ordering is given as follows.

Theorem 5 (Directionally convex maximization).
For $(X,Z) = (X_1,\dots,X_d,Z) \sim \mathcal{EC}_{d+1}(\mu,\Sigma,\phi)$ *, it holds that*

(i) $X \in \mathcal{M}_4$ *, and*

(ii) $\xi \leq_{dcx} X$ *for all* $\xi \in \mathcal{M}_4$ *.*

For $(X',Z') = (X'_1,\dots,X'_d,Z') \in \mathcal{EC}_{d+1}(0,\Sigma,\eta)$ *, define* $Y = (F_i^{-1}(F_{X'_i}(X'_i)))_{1\leq i\leq d}$ *. Then, it holds that*

(iii) $Y \in \mathcal{M}_5$ *, and*

(iv) $\xi \leq_{dcx} Y$ *for all* $\xi \in \mathcal{M}_5$ *.*

Proof. (i): By construction, Σ has rank 2 . Hence, $\phi \in \Phi_{\mathrm{rank}(\Sigma)}$ is fulfilled. Further, it holds that $\sigma_{i,d+1} = \rho_1$ and $\sigma_{j,d+1} = \rho_2$ for $1 \leq i \leq p < j \leq d$. Hence, $X \in \mathcal{M}_4$.

(ii): Let $(\xi, Z'') \sim \mathcal{EC}_d(\mu, S, \psi)$ be an element of $\mathcal{M}_4^f$. Consider an elliptical vector $(\zeta,\zeta_{d+1}) \sim \mathcal{EC}_{d+1}(\mu,\Sigma,\psi)$. Then, it holds that

$$\xi \leq_{sm} \xi^c_{Z''} \leq_{sm} \zeta \leq_{dcx} X\,, \tag{23}$$

where the first inequality follows from Proposition 3. The second inequality is a consequence of Theorem 1 because $S_{d,d} \leq \Sigma_{d,d}$ componentwise, where $S_{d,d}$ and $\Sigma_{d,d}$ denote the restriction of the matrix S and Σ , respectively, to the first d rows and columns. The last equality follows from Corollary 1 using that $R_{2,\psi} \leq_{st} R_{2,\phi}$ and $M(\rho_1,\rho_2) \geq 0$. Altogether, this implies $\xi \leq_{dcx} X$.

(iii): Since η fulfills condition (21), X' has continuous marginal distribution functions, cp. Remark 5(d). Thus, $F_{X'_i}(X'_i) \sim U(0,1)$ is uniformly distributed on $(0,1)$ which implies that $Y_i \sim F_i \in \mathcal{F}_i$. Further, it holds that

$$(X'_i, Z') \sim \mathcal{EC}_2(0,\rho_1,\eta) \quad \text{and} \quad (X'_j,Z') \sim \mathcal{EC}_2(0,\rho_2,\eta)$$

for $1 \leq i \leq p < j \leq d$. This implies that $C_{Y_i,Z'} \in \mathcal{C}^{\rho_1}$ and $C_{Y_j,Z'} \in \mathcal{C}^{\rho_2}$ for $1 \leq i \leq p < j \leq d$. Thus, $(Y,Z') \in \mathcal{M}_5^f$.

(iv): For $(\zeta, Z') \in \mathcal{EC}_{d+1}(0,S,\eta)$, $S \in \mathcal{S}^{\rho_1,\rho_2}$, Proposition 3 and Theorem 1 imply that

$$\zeta \leq_{sm} \zeta^c_{Z'} \leq_{sm} X'\,. \tag{24}$$

By assumption on the generator η , the bivariate vector (X'_i, X'_j) fulfills condition (21) with $\rho = M(\rho_1,\rho_2)$ for $1 \leq i \leq p < j \leq d$. Thus, Lemma 3 implies that (X'_i, X'_j) is conditionally increasing. Since $\mathrm{rank}(\Sigma) = 2$, also the d-variate vector X' is conditionally increasing. As a consequence of (24), the invariance of the supermodular ordering under increasing transformations implies for $(\xi, Z'') \in \mathcal{M}_5^f$, that

$$\xi \leq_{sm} \xi^c_{Z''} \leq_{sm} F_{\xi_i}^{-1}(F_{X'_i}(X'_i)) \leq_{dcx} Y\,, \tag{25}$$

where the last inequality holds true since $F_{\xi_i} \leq_{cx} F_i$ using that the copula of X' is CI, see Müller and Scarsini [25, Theorem 4.5]. Altogether, this implies $\xi \leq_{dcx} Y$. $\square$

Remark 7.

(a) In the class $\mathcal{M}_4$, distributions from different elliptical generators are allowed. The marginal distributions are assumed to be elliptical in order to apply Corollary 1 for the last inequality in (23). Using the transformation invariance by increasing convex functions of the components, more general marginal classes can be assumed for this result to hold true. In the class $\mathcal{M}_5$, the marginal distributions are allowed to come from general classes $\mathcal{F}_i$ of distributions. The elliptical generator for the bivariateg dependence constraints is fixed in order to apply Theorem 1 for the transformed marginals in (25).

(b) By definition of the classes $\mathcal{M}_4$ and $\mathcal{M}_5$, the copulas $C_{\xi_i,Z''}$ of the components of the risk vector $\xi = (\xi_1,\dots,\xi_d)^\top$ with the risk factor Z'' are from the same elliptical generator ψ . Only in this case, the copula $C_{\xi^c_{Z''}}$ of the conditionally comonotonic vector is elliptical (also with generator ψ). For different generators, the conditionally comonotonic random vector is no longer elliptical and, thus, Theorem 1 cannot be applied in the proof of Theorem 5.

Conclusion

In the present paper we derive sufficient criteria and characterizations of the supermodular and of the directionally convex order for the class of elliptical distributions. The results generalize corresponding characterizations for multivariate normal distributions. The proofs are based on conditioning arguments allowing a reduction of the problem to the two- and one-dimensional case.

The ordering results are used in the second part of the paper to derive worst case distributions for several relevant classes of risk models. These include models with elliptical dependence structure and additional bounds on (partial) correlations corresponding to a C-vine structure. We show that these results cannot be generalized to D-vines and, thus, not to arbitrary regular vine structures. A second type of applications concerns partially specified factor models (PSFM), which do not need a full specification of the dependence structure and, thus, are a particular flexible tool for applications. Under various constraints on the specifications, i.e., on the dependence structure of the individual risks with the common risk factor, worst case distributions are determined. As a consequence, these results imply relevant improvements of standard risk bounds based only on marginal information on the risk vectors.

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